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Periodic Schur functions and slit discs ☆

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Dedicated to the memory of G.G. Lorentz

Abstract

A simply connected domain $\mathbb{G}$ is called a slit disc if $\mathbb{G} = \mathbb{D}$ minus a finite number of closed radial slits not reaching the origin. A slit disc is called rational (rationally placed) if the lengths of all its circular arcs between neighboring slits (the arguments of the slits) are rational multiples of 2π. The conformal mapping ϕ of $\mathbb{D}$ onto $\mathbb{G}$, $\phi(0) = 0$, $\phi'(0) > 0$, extends to a continuous function on $\mathbb{T}$ mapping it onto $\partial\mathbb{G}$. A finite union E of closed non-intersecting arcs e_k on $\mathbb{T}$ is called rational if $\nu_E(e_k) \in \mathbb{Q}$ for every k, $\nu_E(e_k)$ being the harmonic measures of e_k at ∞ for the domain $\mathbb{C} \setminus E$. A compact E is rational if and only if there is a rational slit disc $\mathbb{G}$ such that $E = \phi^{-1}(\mathbb{T})$. A compact E essentially supports a measure with periodic Verblunsky parameters if and only if $E = \phi^{-1}(\mathbb{T})$ for a rationally placed $\mathbb{G}$. For any tuple $(\alpha_1, \dots, \alpha_{g+1})$ of positive numbers with $\sum_k \alpha_k = 1$ there is a finite family $\{e_k\}_{k=1}^{g+1}$ of closed non-intersecting arcs e_k on $\mathbb{T}$ such that $\nu_E(e_k) = \alpha_k$. For any set $E = \cup_{k=1}^{g+1} e_k \subset \mathbb{T}$ and any $\epsilon > 0$ there is a rationally placed compact $E^* = \cup_{k=1}^{g+1} e_k^*$ such that the Lebesgue measure $|E \triangle E^*|$ of the symmetric difference $E \triangle E^*$ is smaller than ϵ.

Keywords: Schur's algorithm; Periodic Schur's functions; Wall continued fractions; Wall pairs; Slit domains; Conformal mappings

☆ I am grateful to Leonid Golinskii for a useful discussion during ICM06 in Madrid, which made me believe that slit domains may be related to periodic measures. However it became possible to realize this only after S. Khrushchev, Rational compacts and exposed quadratic irrationalities, J. Approx. Theory (2008), in press (doi:10.1016/j.jat.2009.03.006) was completed.

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1. Slit discs and essential supports of periodic measures

This paper exploits a new relationship between essential supports of measures with periodic Verblunsky parameters and the so-called slit discs. It was motivated by a similar phenomenon in the theory of Hill's operators first found by Marčenko and Ostrovskiĭ in [11]. The discrete version of Marchenko–Ostrovskii theory in the context of orthogonal polynomials on the real line appeared in the paper of Geronimo and Van Assche [5], where slit discs quite similar to those constructed in the present paper were introduced.

A simply connected domain $\mathbb{G}$ is called a *slit disc* if it is obtained from the unit disc $\mathbb{D} = \{z : |z| < 1\}$ by making a finite number $g + 1$ of radial cuts $\mathfrak{l}_k = [r_k\lambda_k, \lambda_k]$, $0 < r_k < 1$, λ_k being the points on the unit circle $\mathbb{T}$ enumerated counterclockwise with the understanding that $\lambda_{g+2} = \lambda_1$. The circular arc $\mathfrak{e}_k = [\lambda_k, \lambda_{k+1}]$ on the boundary $\partial\mathbb{G}$ is placed between two slits $\mathfrak{l}_k$ and $\mathfrak{l}_{k+1}$. In Fig. 1 $\alpha = \mathfrak{e}_1$, $\beta = \mathfrak{e}_2$, $\gamma = \mathfrak{e}_3$, $a = \mathfrak{l}_1$, $b = \mathfrak{l}_2$, $c = \mathfrak{l}_3$. Notice that any slit $\mathfrak{l}_k$ has two sides in $\mathbb{G}$. Any slit disc $\mathbb{G}$ is described by $2g + 2$ parameters, i.e. by $g + 1$ *arguments* λ_k of the slit disc and by $g + 1$ *lengths* r_k of the slits.

By the Riemann mapping theorem [4, Ch IX, Section 1 Theorem 1.2], there is a unique analytic function ϕ, $\phi(0) = 0$, $\phi'(0) > 0$, called the standard conformal mapping, which maps $\mathbb{D}$ one-to-one onto $\mathbb{G}$. Since $\partial\mathbb{G}$ is made of a finite number of smooth curves, the analytic function ϕ extends to a continuous mapping of the closed disc onto the closure $\mathbf{clos}(\mathbb{G})$ of $\mathbb{G}$ by the Schwarz reflection principle; see [4, Ch V, Section 4]. In this particular case there is a formula for ϕ also implying this statement; see (23). Therefore we can consider the inverse images under ϕ of the described parts of $\partial\mathbb{G}$:

$$e_k = \phi^{-1}(\mathfrak{e}_k), \qquad l_k = \phi^{-1}(\mathfrak{l}_k).$$

If $w \in (r_k\lambda_k, \lambda_k)$, then $\phi^{-1}(\{w\})$ consists of two points on $\mathbb{T}$. If $\lambda \in \mathbb{T}$ and ϕ is the standard conformal mapping onto $\mathbb{G}$, then $z \to \lambda\phi(\bar{\lambda}z)$ is the standard conformal mapping of $\mathbb{D}$ onto the rotated slit disc $\lambda\mathbb{G}$. Notice that if $\mathbb{G}$ rotates counterclockwise then the inverse image of the boundary $\partial\mathbb{G}$ on $\mathbb{T}$ also rotates counterclockwise. This can be easily seen from the equivalence

$$\lambda\phi(\bar{\lambda}z) \in \lambda\mathfrak{l}_k \Leftrightarrow z \in \lambda l_k.$$

Definition 1.1. A slit disc is called rational (rationally placed) if the lengths of all its circular arcs between neighboring slits (the arguments of the slits) are rational multiples of 2π.

A probability measure $\sigma \in \mathfrak{P}(\mathbb{T})$ on $\mathbb{T}$ is called periodic if its Schur's function $f = f^\sigma$ defined by

$$\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, \mathrm{d}\sigma(\zeta) = \frac{1 + zf^\sigma}{1 - zf^\sigma}, \tag{1}$$

generates in the Hardy algebra H^∞ a periodic Schur's algorithm:

$$f_0 = \frac{zf_1(z) + a_0}{1 + \bar{a}_0 zf_1(z)}; \ldots; f_n = \frac{zf_{n+1}(z) + a_n}{1 + \bar{a}_n zf_{n+1}(z)}; \ldots. \tag{2}$$

The sequence $\mathcal{S}(f) = \{a_n\}_{n\geqslant 0}$, $a_n = f_n(0)$, is called the sequence of the *Schur parameters* of f. Since by (1) the correspondence $\sigma \longleftrightarrow f = f^\sigma$ is one-to-one, the numbers $\{a_n\}_{n\geqslant 0}$ can be considered as parameters of σ. By an old Geronimus theorem [6,7,14], they coincide with the Verblunsky parameters of σ. Since Schur's algorithm (2) is uniquely determined by its

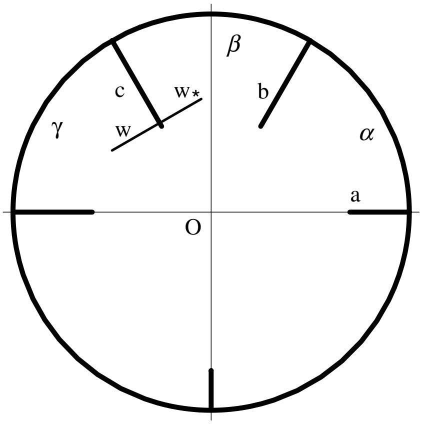

Fig. 1. A slit disc.

parameters $\{a_n\}_{n\geqslant 0}$, a measure σ in $\mathfrak{P}(\mathbb{T})$ (a Schur's function $f = f^\sigma$) is periodic if and only if $\{a_n\}_{n\geqslant 0}$ is periodic and is pure periodic if $\{a_n\}_{n\geqslant 0}$ is pure periodic.

By Theorem 1.2 of [8] the essential support $\mathbf{supp}_{ess}(\sigma)$ of any periodic measure σ is the closure E of the set $\mathcal{E}(f^\sigma) = \{t \in \mathbb{T} : |f^\sigma(t)| < 1\}$:

$$E = \bigcup_{k=1}^{g+1} e_k, \tag{3}$$

where $\{e_k\}_{k=1}^{g+1}$ is a finite family of closed non-intersecting arcs on $\mathbb{T}$. The main result of the present paper claims that the essential supports of periodic measures are the inverse images under standard conformal mappings of the circular parts of the boundaries of rationally placed slit discs.

Theorem 1.1. *A closed subset E of the form (3) is the essential support of a periodic measure if and only if there exists a rationally placed slit disc $\mathbb{G}_E$ with slits $\mathfrak{l}_k$ and arcs $\mathfrak{e}_k$ such that its standard conformal mapping ϕ satisfies*

$$e_k = \phi^{-1}(\mathfrak{e}_k), \quad k = 1, \ldots, g+1.$$

2. Wall pairs and slit discs

Since $\mathcal{E}(f)$ is invariant under Schur's algorithm (see [9] or [8,10]), the families of essential supports of periodic and pure periodic measures coincide. The Schur function $f = f^\sigma$ of any non-zero pure periodic $\sigma \in \mathfrak{P}(\mathbb{T})$ with period $d+1$ is uniquely determined by a Wall pair (A, B) of relatively prime polynomials A and B:

$$f = \frac{A + zB^*f}{B + zA^*f} \Longleftrightarrow zA^*f^2 + (B - zB^*)f - A = 0. \tag{4}$$

The degree $d = \deg(A)$ of A is called the degree of the Wall pair. A pair (A, B) of polynomials is a Wall pair of degree d if and only if $B(0) = 1$, B has no zeros λ with $|\lambda| \leqslant 1$, $\deg(A) = d$, and

$$|B|^2 - |A|^2 = \omega > 0 \quad \text{on } \mathbb{T}; \tag{5}$$

see [9, Theorem 2.1]. Multiplying (5) by z^d, we obtain an important formula

$$B^*B - A^*A = \omega z^d, \tag{6}$$

where

$$A^* = z^d\overline{A(1/\bar{z})}, \;\; B^* = z^d\overline{B(1/\bar{z})}.$$

Formula (6) is called the *determinant identity*. By (6) and (4)

$$f - \frac{A}{B} = \frac{(zB^*B - zA^*A)f}{B(B + zA^*f)} = \frac{\omega z^{d+1} f}{B(B + zA^*f)},$$

implying the periodicity of f. Any Wall pair (A, B) is uniquely determined by the polynomial A, $\deg(A) = d$, and the parameter ω. The only restriction on A is the inequality

$$\int_{\mathbb{T}} \log|A|^2 \, \mathrm{d}m < 0. \tag{7}$$

Then there is a unique $\omega \in (0, 1)$ such that

$$\int_{\mathbb{T}} \log(|A|^2 + \omega) \, \mathrm{d}m = 0. \tag{8}$$

By Fejér's theorem and Jenssen's formula (see [15]), the equation $|B|^2 = |A|^2 + \omega$ has a solution in polynomials with $B(0) = 1$. Such a B cannot vanish in $\mathbb{D}$.

Every Wall pair (A, B) determines an important entire algebraic function

$$\sqrt{\omega}X^2 - b(z)X + \sqrt{\omega}z^{d+1} = 0, \tag{9}$$

where $b = B + zB^*$, $\deg B = d$ is a polynomial with roots on $\mathbb{T}$ placed exactly at the solutions to the equation $zB^*/B = -1$. If f is a Schur function satisfying (4), then by [8, Theorem 1.2] the closure E of

$$\mathcal{E}(f) = \{t \in \mathbb{T} : |b|^2 - 4\omega < 0\} \tag{10}$$

is the essential support of σ with the Schur function f. The set E has $g + 1$ components as indicated in (3). On the one hand $g \leqslant d$; see [8, Corollary 5.12]. On the other hand the method of slit discs presented in this paper shows that given any pair $g \leqslant d$ one can construct a set E with $g + 1$ components which is an essential support of a periodic measure of period $d + 1$; see Corollary 5.5.

Eq. (9) has two solutions

$$\rho_1(z) = \frac{\sqrt{\mathcal{D}}}{2\sqrt{\omega}}\left\{\frac{b}{\sqrt{\mathcal{D}}} + 1\right\}, \qquad \rho_2(z) = \frac{\sqrt{\mathcal{D}}}{2\sqrt{\omega}}\left\{\frac{b}{\sqrt{\mathcal{D}}} - 1\right\}, \tag{11}$$

where $\rho_1(0) = \omega^{-1/2}$ and $\mathcal{D} = b^2 - 4\omega z^{d+1}$ is the discriminant of (9). Since by Viète's formula $\rho_1\rho_2 = z^{d+1}$, the function ρ_1 does not vanish in the closed unit disc. Since ρ_2/ρ_1 is a contractive analytic function in $\mathbb{D}$ by [8, Theorem 4.3, (76)],

$$\phi = \left(\frac{\rho_2}{\rho_1}\right)^{\frac{1}{d+1}}, \qquad \phi(0) > 0, \tag{12}$$

also is analytic and contractive in $\mathbb{D}$.

Let us consider the simplest case of 1-periodic measures. Then $B = 1$ and $A = a$, $a \in \mathbb{D}$. We assume for simplicity that $a > 0$. Hence $\omega = 1 - a^2$ and

$$b(z) = 1 + z = 2e^{i\frac{\theta}{2}} \cos\frac{\theta}{2};$$
$$\mathcal{D} = (1+z)^2 - 4(1-a^2)z = 4e^{i\theta}\left\{\cos^2\frac{\theta}{2} - (1-a^2)\right\}, z = e^{i\theta}.$$

Since both b and $\mathcal{D}$ are positive on $[0, 1]$, we obtain that on $\mathbb{T}$

$$\frac{b}{\sqrt{\mathcal{D}}} = \frac{\cos\frac{\theta}{2}}{\sqrt{a^2 - \sin^2\frac{\theta}{2}}} = \frac{\cos\frac{\theta}{2}}{\sqrt{\cos^2\frac{\theta}{2} - \omega}},$$

where $\sqrt{x} > 0$ if $x > 0$. Hence $b/\sqrt{\mathcal{D}} > 1$ on the open circular arc l from $-2\arcsin a$ to $+2\arcsin a$ with center at $z = 1$ and $b/\sqrt{\mathcal{D}}$ is pure imaginary on its complement $\Delta_a = \mathbb{T} \setminus l$. Since $b/\sqrt{\mathcal{D}}$ is a function of $\cos\frac{\theta}{2}$, it is even on $(-2\arcsin a, 2\arcsin a)$ and monotonically increases on $[0, 2\arcsin a)$ from $1/a$ to $+\infty$. It follows that

$$\phi(e^{i\theta}) = \frac{b/\sqrt{\mathcal{D}} - 1}{b/\sqrt{\mathcal{D}} + 1} = \frac{4\omega z}{\left(z + 1 + \sqrt{(z+1)^2 - 4\omega z}\right)^2}, \tag{13}$$

and $\phi(e^{i\theta})$ moves from $(1-a)/(1+a)$ to 1 inside the slit $[(1-a)/(1+a), 1]$ when θ moves from 0 to $2\arcsin a$. Starting from $\theta = 2\arcsin a$ the point $\phi(e^{i\theta})$ moves anticlockwise along $\mathbb{T}$ until it returns back to 1 at $\theta = 2\pi - 2\arcsin a$. By the symmetry it moves back to $(1-a)/(1+a)$ along the slit $[(1-a)/(1+a), 1]$ when θ increases from $2\pi - 2\arcsin a$ to 2π. By the argument principle ϕ is the standard conformal mapping of $\mathbb{D}$ onto the open unit disc with a slit along the segment of the real line $[(1-a)/(1+a), 1]$; see [4, Theorem 1.3]. One may observe that

$$\phi(z) = \frac{z}{\Phi^2(z)}, \tag{14}$$

where

$$\Phi(z) = \frac{z + 1 + \sqrt{(z+1)^2 - 4\omega z}}{2\sqrt{\omega}}$$

is the conformal mapping of $\mathbb{C} \setminus \Delta_a$ onto the domain $\{|z| > 1\}$ such that $\Phi(z) \sim \omega^{-1/2} z$ as $z \to \infty$ [7, Lemma 8.121]. We are going to extend the important formula (14) to arbitrary Wall pairs.

3. The multi-valued function Φ

It is well-known (see e.g. [13]) that for a finite union $E = \cup_{k=1}^{g+1} e_k$ of closed non-intersecting arcs on $\mathbb{T}$ the Green's function $g_E(z)$ is uniquely defined as a positive harmonic function in $\mathbb{C} \setminus E$ which is continuous at every point of E, vanishes on E and satisfies

$$g_E(z) = \log|z| + \log\frac{1}{C(E)} + o(1), \quad \text{as } z \to \infty.$$

The constant $C(E) \in (0, 1)$ is called the *logarithmic capacity* of E.

In a simply connected neighborhood of every $z_0 \in \mathbb{C} \setminus E$ there exists a harmonic function $\tilde{g}_E(z)$, such that $g_E(z) + \mathrm{i}\tilde{g}_E(z)$ is analytic about z_0. The function $\tilde{g}_E(z)$ is called *harmonically conjugate* to $g_E(z)$. The Cauchy–Riemann equations for $g_E(z)+\mathrm{i}\tilde{g}_E(z)$ determine $\tilde{g}_E(z)$ locally up to a constant. We fix the choice of $\tilde{g}_E(z)$ about $z = 0$ by the requirement $\tilde{g}_E(0) = 0$. The function $g_E(z)+\mathrm{i}\tilde{g}_E(z)$ extends from the unit disc $\mathbb{D}$ to a multi-valued analytic function in $\mathbb{C}\setminus E$. If z makes a loop in $\mathbb{C} \setminus E$ then $g_E(z) + \mathrm{i}\tilde{g}_E(z)$ returns back to the starting point of the loop with a pure imaginary increment. Following [2,16] and [17] we associate with each E a multi-valued function analytic in $\mathbb{C} \setminus E$ with the single-valued modulus:

$$\Phi(z) = \Phi_E(z) = \exp\{g_E(z) + \mathrm{i}\tilde{g}_E(z)\}. \tag{15}$$

If E is fixed and there is no confusion we denote by Φ_1 the single-valued branch of Φ_E in $\mathbb{D}$ such that $\Phi_1(0) > 1$.

The open set $\mathbb{T}\setminus E$ in $\mathbb{T}$ is made of $g+1$ open circular arcs l_k, which are enumerated so that e_k follows l_k counterclockwise. The standard stereographic projection of the unit sphere S^2 in $\mathbb{R}^3$ placed with its southern pole at the origin above the coordinate plane XOY maps the northern pole of S^2 to the infinity; see e.g. [1]. We denote by e^- the side of a circular arc $e \subset \mathbb{T}$ facing ∞ and by e^+ its side facing 0. Similarly f^+ are the boundary values on $\mathbb{T}$ of a function f from the side of 0 and f^- are the boundary values from the side of ∞.

The function Φ_1 has a number of important properties. To begin with, by (15) it satisfies

$$\frac{\overline{\Phi_1^+(\zeta)}}{\Phi_1^+(\zeta)} = \exp\{-2\mathrm{i}\tilde{g}_E^+(\zeta)\}, \quad \zeta \in \mathbb{T}. \tag{16}$$

By [8, (19)] the radial limits $\Phi_1^+(\zeta)$ of Φ_1 on $\mathbb{T} \setminus E$ satisfy

$$\frac{\overline{\Phi_1^+(\zeta)}}{\Phi_1^+(\zeta)} = \lambda_k\bar{\zeta}, \quad \zeta \in l_k. \tag{17}$$

The numbers λ_k are called in [8] the phases of Φ_E. By (16) and (17)

$$2\tilde{g}_E^+(\mathrm{e}^{\mathrm{i}\theta}) = \theta - \arg(\lambda_k), \quad \mathrm{e}^{\mathrm{i}\theta} \in l_k. \tag{18}$$

The multi-valued function $g_E(z) + \mathrm{i}\tilde{g}_E(z)$ is analytic everywhere except for the ends $z = t_j^\pm$ of the connected components e_j of E. We assume that t_j^+ follows t_j^- counterclockwise so that $e_j = [t_j^-, t_j^+]$. Therefore we may consider continues branches of $\tilde{g}_E^+$ and $\tilde{g}_E^-$ on $e_j = [t_j^-, t_j^+]$. Since e_j is connected, any two such branches differ by a constant. The following lemma follows from the Cauchy–Riemann equations; see [8, Lemma 3.6].

Lemma 3.1. *On every e_k the function $\tilde{g}_E^+$ decreases counterclockwise and $\tilde{g}_E^-$ increases counterclockwise. In particular $\tilde{g}_E^- - \tilde{g}_E^+$ increases counterclockwise. The sum $\tilde{g}_E^- + \tilde{g}_E^+$ equals $\theta + \delta_k$, δ_k being a real constant.*

Proof. Since $g_E \equiv 0$ on E, the second Cauchy–Riemann equation implies that $\partial\widetilde{g_E}/\partial n^\pm \equiv 0$ on (t_j^-, t_j^+). Next,

$$0 \leqslant \frac{\partial g_E}{\partial n^+}(\mathrm{e}^{\mathrm{i}\theta}) = -\lim_{s\to 1^-} \frac{\partial\widetilde{g}_E}{\partial\theta}(s\mathrm{e}^{\mathrm{i}\theta}) = -\frac{\partial\widetilde{g}_E^+}{\partial\theta}(\mathrm{e}^{\mathrm{i}\theta}),$$

$$0 \leqslant \frac{\partial g_E}{\partial n^-}(\mathrm{e}^{\mathrm{i}\theta}) = \lim_{s\to 1^+} \frac{\partial\widetilde{g}_E}{\partial\theta}(s\mathrm{e}^{\mathrm{i}\theta}) = \frac{\partial\widetilde{g}_E^-}{\partial\theta}(\mathrm{e}^{\mathrm{i}\theta}).$$

By [8, Lemma 2.1]

$$\frac{\partial g_E}{\partial n^-} = 1 + \frac{\partial g_E}{\partial n^+}, \tag{19}$$

implying the last statement in the lemma. □

4. The conformal mapping ϕ_E

By the Green's formula and Lemma 3.1 the harmonic measures $\nu_E(e_k)$ satisfy

$$2\pi\nu_E(e_k) = \int_{e_k^+} \frac{\partial g_E}{\partial n^+}|\mathrm{d}\zeta| + \int_{e_k^-} \frac{\partial g_E}{\partial n^-}|\mathrm{d}\zeta| = \overleftarrow{\underset{e_k^+}{\Delta}}(\tilde{g}_E^- - \tilde{g}_E^+). \tag{20}$$

The phases λ_k of Φ_E satisfy

$$\lambda_{k+1} = \lambda_k e^{2\pi i \nu_E(e_k)}, \quad k = 1, \ldots, g+1, \qquad \lambda_{g+2} = \lambda_1; \tag{21}$$

see [8, Lemma 3.7], or combine (18) with (20). For $k = 1, \ldots, g+1$ we put

$$r_k = \inf_{\zeta \in l_k} |\Phi_E(\zeta)|^{-2}. \tag{22}$$

It is clear that $0 < r_k < 1$. Now we make $g+1$ radial cuts $\mathfrak{l}_k = [r_k\lambda_k, \lambda_k]$ in $\mathbb{D}$ and denote by $\mathbb{G}_E$ the slit disc obtained. By (21) the circular distance between neighboring slits of $\mathbb{G}_E$ is $2\pi\nu_E(e_k)$. By [8, (17)]

$$\phi_E(z) = \frac{z}{\Phi_E^2(z)} = z\,C(E)^2 \exp \int_0^z \left\{ \frac{r_E(\zeta)}{\zeta\sqrt{\mathcal{D}_E(\zeta)}} - \frac{1}{\zeta} \right\} \mathrm{d}\zeta, \quad |z| \leqslant 1, \tag{23}$$

where

$$\begin{aligned} r_E(z) &= -(z-\tau_1)\cdots(z-\tau_{g+1}), \\ \mathcal{D}_E(z) &= \prod_{k=1}^{g+1}(z-t_k^-)(z-t_k^+). \end{aligned} \tag{24}$$

In (24) τ_k are the zeros of the harmonic conjugate function $\tilde{\nu}_E$ on $\mathbb{T}$ for the equilibrium measure ν_E of the compact E. Each open interval l_k contains exactly one zero τ_k.

Corollary 4.1. *For every ϕ_E we have*

$$\phi_E'(0) = C(E)^2. \tag{25}$$

Proof. Differentiate (23) and put $z = 0$. □

Theorem 4.2. *For any $E = \cup_{k=1}^{g+1} e_k$ the function*

$$\phi_E(z) = \frac{z}{\Phi_E^2(z)} \tag{26}$$

is the standard conformal mapping of $\mathbb{D}$ onto the slit disc $\mathbb{G}_E$.

Proof. By the Schwarz reflection principle [4, Ch V, Section 4] or by (23) ϕ_E extends to a continuous function in the closed unit disc. If $\zeta \in l_k$, then (17) shows that

$$\phi_E(\zeta) = \frac{1}{|\Phi_E|^2}\zeta\frac{\overline{\Phi_E(\zeta)}}{\Phi_E(\zeta)} = \frac{\lambda_k}{|\Phi_E|^2},$$

implying that $\phi_E(\zeta)$ maps l_k onto the cut $\mathfrak{l}_k$. Combining this formula with (23), we obtain for $z \in l_k$ that

$$\begin{aligned}|\phi_E(z)| &= \frac{1}{|\Phi_E(z)|^2} = C\exp\mathfrak{Re}\left(\int_{t_k^+}^{z}\left\{\frac{r_E(\zeta)}{\zeta\sqrt{\mathcal{D}_E(\zeta)}} - \frac{1}{\zeta}\right\}\mathrm{d}\zeta\right)\\ &= C\exp\mathfrak{Re}\left(\mathrm{i}\int_{t_k^+}^{z}\left\{\frac{r_E(\zeta)}{\sqrt{\mathcal{D}_E(\zeta)}} - 1\right\}\mathrm{d}\theta\right)\\ &= C\exp\mathfrak{Re}\left(\mathrm{i}\int_{t_k^+}^{z}\frac{r_E(\zeta)}{\sqrt{\mathcal{D}_E(\zeta)}}\mathrm{d}\theta\right) = C\exp\left(\int_{t_k^+}^{z}\frac{\mathrm{i}r_E(\zeta)}{\sqrt{\mathcal{D}_E(\zeta)}}\mathrm{d}\theta\right),\end{aligned}$$

since $r_E(z)\mathcal{D}_E(z)^{-1/2}$ is imaginary on $\mathbb{T}\setminus E$; see [8, Theorem 3.2]. It follows that

$$\frac{\mathrm{d}|\phi_E(\mathrm{e}^{\mathrm{i}\theta})|}{\mathrm{d}\theta} = |\phi_E(\mathrm{e}^{\mathrm{i}\theta})|\frac{\mathrm{i}r_E(\mathrm{e}^{\mathrm{i}\theta})}{\sqrt{\mathcal{D}_E(\mathrm{e}^{\mathrm{i}\theta})}},$$

vanishes only at the zeros τ_k of r_E on l_k. Hence when ζ moves from t_{k-1}^+ to t_k^- its image $\phi_E(\zeta)$ first moves to the center along $\mathfrak{l}_k$ until ζ reaches τ_k. After that it returns back to the circle.

If $\zeta = \mathrm{e}^{\mathrm{i}\theta}$ moves counterclockwise along e_k, then $\phi_E(\mathrm{e}^{\mathrm{i}\theta}) = \mathrm{e}^{\mathrm{i}(\theta - 2\tilde{g}_E^+)}$. Since

$$\theta - 2\tilde{g}_E^+ = \tilde{g}_E^- + \tilde{g}_E^+ - 2\tilde{g}_E^+ - \delta_k = \tilde{g}_E^- - \tilde{g}_E^+ - \delta_k$$

increases by Lemma 3.1, the argument of $\phi_E(\zeta)$ increases by $2\pi\nu_E(e_k)$; see (21). Since the argument of ϕ_E monotonically increases by 2π along $\mathbb{T}$, the argument principle implies that ϕ_E is a conformal mapping of $\mathbb{D}$ onto $\mathbb{G}_E$ [4, Theorem 1.3]. $\square$

Theorem 4.3 (*Bieberbach–Köbe*). *Let G be the image of* $\mathbb{D}$ *under the conformal mapping*

$$w = f(z) = z + a_2z^2 + \cdots + a_nz^n + \cdots.$$

Then G always contains the disc $\{w : |w| < \frac{1}{4}\}$.

The proof of this theorem can be found in for instance [12, Theorem 1.6, p 16–17$_3$]. Combining Theorem 4.2 and formula (23) with the Bieberbach–Köbe theorem, we can easily find a relationship of $C(E)$ with the radii r_k of the slit discs G_E.

Corollary 4.4. *Let* $E \subset \mathbb{T}$ *be a closed subset of* $\mathbb{T}$ *with a finite number of connected components. Let* ϕ_E *be the conformal mapping of* $\mathbb{D}$ *onto a slit disc* $\mathbb{G}_E$ *with the slits* $\mathfrak{l}_k = [r_k\lambda_k, \lambda_k]$ *in* $\mathbb{D}$. *Then*

$$C(E)^2 \leqslant \min_k 4r_k. \tag{27}$$

Proof. By the Bieberbach–Köbe theorem the image $C(E)^{-2}\mathbb{G}_E$ of $\mathbb{D}$ under the conformal mapping $z \to C(E)^{-2}\phi_E(z)$ contains the disc $\{|z| \leqslant 0.25\}$. $\square$

By Corollary 4.4 the set E has small capacity if at least one slit $\mathfrak{l}_k$ of $\mathbb{G}_E$ goes too close to the origin. Another theorem [12, Theorem 1.5, p 15_3] estimates the coefficient at z^2 in the Maclaurin series of $C(E)^{-2}\phi_E$.

Theorem 4.5. *Let G be the image of $\mathbb{D}$ under the conformal mapping*

$$w = f(z) = z + a_2 z^2 + \cdots + a_n z^n + \cdots.$$

Then $|a_2| \leqslant 2$.

Since $(r_E/\sqrt{\mathcal{D}})(0) = 1$ by (23),

$$C(E)^{-2}\phi_E(z) = z + \left(\frac{r_E}{\sqrt{\mathcal{D}}}\right)'(0)z^2 + \cdots.$$

Simple differentiation shows that

$$\left(\frac{r_E}{\sqrt{\mathcal{D}}}\right)' = \frac{r_E'}{\sqrt{\mathcal{D}}} - \frac{r_E}{\sqrt{\mathcal{D}}}\frac{\mathcal{D}'}{2\mathcal{D}} = \frac{r_E}{\sqrt{\mathcal{D}}}\left(\frac{r_E'}{r_E} - \frac{\mathcal{D}'}{2\mathcal{D}}\right).$$

By Theorem 4.5 the sum of the distances from the middles of the chords completing arcs l_k to the zeros τ_k of r_E in l_k cannot exceed 2:

$$\left|\tau_k - \sum_{k=1}^{g+1}\frac{t_k^+ + t_{k+1}^-}{2}\right| = \left|\left(\frac{r_E'}{r_E} - \frac{\mathcal{D}'}{2\mathcal{D}}\right)(0)\right| \leqslant 2. \tag{28}$$

The conformal mapping ϕ_E extends by the Schwarz reflection principle through $E = \cup_{k=1}^{g+1} e_k$ to the conformal mapping of the domain $\hat{\mathbb{C}}$ with the circular slits l_k onto the domain $\hat{\mathbb{C}}$ with the radial slits $[r_k\lambda_k, r_k^{-1}\lambda_k]$.

Theorem 4.6. *Let ϕ be the conformal mapping of $\mathbb{D}$ onto a slit disc $\mathbb{G}$ with the radial slits $\{\mathfrak{l}_k\}_{k=1}^{g+1}$ and the arcs $\{\mathfrak{e}_k\}_{k=1}^{g+1}$ such that $\phi(0) = 0$, $\phi'(0) > 0$. Let $E = \cup_{k=1}^{g+1} e_k$, where $e_k = \phi^{-1}(\mathfrak{e}_k)$. Then*

$$\Phi_E = \Phi_1(z) = \sqrt{\frac{z}{\phi(z)}}, \quad \sqrt{1} = 1. \tag{29}$$

Proof. Let $\{l_k\}_{k=1}^{g+1}$ be the set of open complementary arcs for E. Then

$$\phi(\mathrm{Clos}\,(l_k)) = \mathfrak{l}_k.$$

Since l_k is an arc and $\mathfrak{l}_k$ is a segment of line, the analytic function ϕ extends by the symmetry to $\{|z| > 1\}$. Namely if $\phi(z) = w$, then $\phi(1/\bar{z}) = w^*$, where w^* is the symmetric reflection of w with respect to the line passing through $\mathfrak{l}_k$; see Fig. 1. Since $|w^*| = |w|$ whatever slit $\mathfrak{l}_k$ is taken, the extension of ϕ to $\mathbb{C} \setminus E$ through $\cup_{k=1}^{g+1} l_k$ defines in $\mathbb{C} \setminus E$ a multi-valued analytic function with the single-valued modulus. Hence the function

$$g(z) = \frac{1}{2}\log\frac{|z|}{|\phi(z)|}$$

is harmonic in $\mathbb{C} \setminus E$, equals 0 on E and is positive in $\mathbb{C} \setminus E$. Since $|\phi(z)| = |\phi(1/\bar{z})|$ and $\phi(z) \sim \phi'(0)z$, $z \to 0$, we conclude that

$$|\phi(z)| = |\phi(1/\bar{z})| \sim \frac{\phi'(0)}{|z|}, \quad z \to \infty.$$

It follows that $g(z) = \log|z| + O(1)$ as $z \to \infty$. This implies that $g = g_E$ is the Green's function for $\mathbb{C} \setminus E$ proving the theorem. $\square$

Corollary 4.7. *Let ϕ, $\phi(0) = 0$, $\phi'(0) > 0$ be the standard conformal mapping of $\mathbb{D}$ onto a slit disc G with the radial slits $\{\mathfrak{l}_k\}_{k=1}^{g+1}$ and the arcs $\{\mathfrak{e}_k\}_{k=1}^{g+1}$ on $\mathbb{T}$. Let $e_k = \phi^{-1}(\mathfrak{e}_k)$ and $E = \cup_{k=1}^{g+1} e_k$. Then*

$$|\mathfrak{e}_k| = 2\pi \nu_E(e_k). \tag{30}$$

Corollary 4.8. *For any tuple $(\alpha_1, \dots, \alpha_{g+1})$ of positive numbers with $\sum_k \alpha_k = 1$ there is a finite family $\{e_k\}_{k=1}^{g+1}$ of closed non-intersecting arcs e_k placed on $\mathbb{T}$ counterclockwise such that $\nu_E(e_k) = \alpha_k$.*

Proof. Since $\sum_k \alpha_k = 1$, there is a family of arcs $\{\mathfrak{e}_k\}_{k=1}^{g+1}$ enumerated in the counterclockwise direction covering $\mathbb{T}$. Let $\mathbb{G}$ be obtained from $\mathbb{D}$ by making cuts $\mathfrak{l}_k$ of arbitrary length $r_k \in (0, 1)$ at the ends of e_k. Let ϕ be the standard conformal mapping of $\mathbb{D}$ onto $\mathbb{G}$. By (21) $\nu_E(e_k) = \alpha_k$, where $e_k = \phi^{-1}(\mathfrak{e}_k)$. $\square$

Corollary 4.9. *For every finite union $E = \cup_{k=1}^{g+1} e_k$ of closed non-intersecting arcs on $\mathbb{T}$ enumerated counterclockwise there is a unique slit disc $\mathbb{G}$ such that the standard conformal mapping ϕ_E maps $\mathbb{D}$ onto $\mathbb{G}$ and $E = \phi_E^{-1}(\mathbb{T})$.*

Proof. Apply Theorem 4.2. $\square$

5. Rational compacts

A $\lambda \in \mathbb{T}$ is called unitary rational if $\lambda^n = 1$ for some $n \in \mathbb{Z}$. A compact $E = \cup_{k=1}^{g+1} e_k$ is called *rational* if $\nu_E(e_k) \in \mathbb{Q}$ for every k. The smallest positive number $P = P(E)$ such that $P\nu_E(e_k) \in \mathbb{Z}$ for every k is called the *period* of a rational compact E. A rational compact E is called *rationally placed* if all phases of Φ_E are unitary rational. A rationally placed compact E is called *balanced* if $\lambda_k^P = 1$ for every phase λ_k of Φ_E, where $P = P(E)$; see [8, Definition 1.5].

If b is a polynomial with roots on $\mathbb{T}$ of degree $d + 1$, then $|b|$ has $d + 1$ local maxima on $\mathbb{T}$, the smallest of which we denote by m_b. An elementary proof of the following theorem can be found in [8, Theorem 6.6].

Theorem 5.1. *A compact E is rational if and only if there are a monic polynomial b with roots on $\mathbb{T}$ and $0 < 4\omega \leqslant m_b^2$ such that $E = \{t \in \mathbb{T} : |b(t)|^2 \leqslant 4\omega\}$.*

For rational compacts the function Φ_E is algebraic and can be represented as

$$\varrho_1(z) = \Phi^P(z), \qquad \varrho_2(z) = \overline{\Phi}(1/\bar{z})^{-P}. \tag{31}$$

Here ϱ_1 and ϱ_2 are the roots of the following quadratic equation:

$$\begin{aligned} &\sqrt{\omega}X^2 - b_E(z)X + \sqrt{\omega}z^P = 0, \\ &\sqrt{\omega} = C(E)^P = \varrho_1(0)^{-1}, \qquad b_E = \varrho_1(0)^{-1}(\varrho_1 + \varrho_2), \end{aligned} \tag{32}$$

b_E being a monic polynomial of degree P with simple roots on E. We denote by $\mathcal{D} = b_E^2 - 4\omega z^P$ the discriminant of (32). By [8, Theorem 4.3]:

Theorem 5.2. *Any rational compact E on $\mathbb{T}$ is the closure of*

$$\mathcal{E} = \{t \in \mathbb{T} : |b_E|^2 - 4\omega < 0\}, \tag{33}$$

where $b_E = \varrho_1(0)^{-1}(\varrho_1 + \varrho_2)$, is a separable polynomial of degree $P = P(E)$. The function $b_E\mathcal{D}^{-1/2}$ is positive on $\mathbb{T} \setminus E$, is pure imaginary on E, and

$$\frac{1 + \varrho_2/\varrho_1}{1 - \varrho_2/\varrho_1}(z) = \frac{b_E}{\sqrt{\mathcal{D}}}(z), \tag{34}$$

where the branch of $\sqrt{\mathcal{D}}$ is chosen so that $\sqrt{\mathcal{D}(0)} = \sqrt{1} = 1$ is a function with positive real part in $\mathbb{D}$.

By (31), (32) and (34) on the unit circle

$$\frac{1}{|\Phi_E|^{2P}} = \left|\frac{z^P}{\varrho_1^2}\right| = \left|\frac{\varrho_1\varrho_2}{\varrho_1^2}\right| = \left|\frac{\varrho_2}{\varrho_1}\right| = \left|\frac{b/\sqrt{\mathcal{D}} - 1}{b/\sqrt{\mathcal{D}} + 1}\right|. \tag{35}$$

Theorem 5.2 and the method of slit discs lead to a natural parametrization of rational compacts. An n-tuple $(\alpha)_n$ of positive numbers $(\alpha_1, \ldots, \alpha_n)$ such that $\sum_k \alpha_k = 1$ is called rational if $\alpha_k \in \mathbb{Q}$ for every k. A positive integer $d + 1$ is called the period of $(\alpha)_n$ if it is the smallest positive integer such that $(d + 1)\alpha_k \in \mathbb{Z}$ for every k.

Theorem 5.3. *Let $g + 1$ be a positive integer, $(x)_{g+1}$ be a $g + 1$-tuple of positive real numbers such that $x_k > 4$ for every k and $(\alpha)_{g+1}$ be a rational $g + 1$-tuple with period $d + 1$. Then there exist a unique up to a rotation monic polynomial b, $\deg(b) = d + 1$, with all roots on $\mathbb{T}$ and $\omega \in (0, 1)$ such that:*

(a) $\{t \in \mathbb{T} : |b|^2 > 4\omega\} = \cup_{k=1}^{g+1} l_k$, *where l_k are open arcs on $\mathbb{T}$ with non-intersecting closures;*
(b) $\max_{t \in l_k} |b|^2 = x_k\omega$ *for every k;*
(c) $\nu_E(e_k) = \alpha_k$ *for every k, e_k being the closed arcs between l_k and l_{k+1}.*

Proof. Since

$$x \longrightarrow \frac{1}{x} + 2 + x$$

decreases on $(0, 1)$ from infinity to 4 there is a unique $g + 1$ tuple $(r_1, r_2, \ldots, r_{g+1})$ of numbers in $(0, 1)$ such that

$$\frac{(1 + r_k^{d+1})^2}{r_k^{d+1}} = x_k, \quad k = 1, \ldots, g + 1.$$

Let $\mathbb{G}$ be the slit disc obtained from $\mathbb{D}$ by $g + 1$ radial cuts $\mathfrak{l}_k$ of lengths $1 - r_k$ separated by circular arcs $\mathfrak{e}_k$, $|\mathfrak{e}_k| = 2\pi\alpha_k$, and ϕ be the standard conformal mapping of $\mathbb{D}$ onto $\mathbb{G}$. Then by

Corollary 4.8 $E = \phi^{-1}(\cup_{k=1}^{g+1} \mathfrak{e}_k)$ is a rational compact on $\mathbb{T}$ with period $P = d+1$. Hence by Theorem 5.2 there are a polynomial $b = b_E$, $\deg(b) = d+1$, with roots on E and $\omega \in (0,1)$ such that

$$\{t \in \mathbb{T} : |b|^2 - 4\omega \leqslant 0\} = E.$$

Let $l_k = \mathfrak{Int}(\phi^{-1}(\mathfrak{l}_k))$. By Theorem 5.2 ϱ_2/ϱ_1 and $b\sqrt{\mathcal{D}}$ are positive on l_k. By (35) and the definition of r_k (see (22)), we have

$$r_k^{d+1} = \inf_{\zeta \in l_k} |\Phi_E(\zeta)|^{-2d-2} = \inf_{l_k} \frac{\rho_2}{\rho_1} = \frac{\inf_{l_k} b/\sqrt{D} - 1}{\inf_{l_k} b/\sqrt{D} + 1}$$
$$= 1 - \frac{2}{1 + \inf_{l_k} b/\sqrt{D}} = 1 - \frac{2}{1 + \frac{\max_{l_k} |b|}{\sqrt{\max_{l_k} |b|^2 - 4\omega}}}.$$

Resolving the above equation in $\max_{l_k} |b|$, we complete the proof of (b). Finally (c) follows from (30). The uniqueness up to a rotation follows from the uniqueness of the standard conformal mapping. $\square$

The polynomial b constructed in Theorem 5.3 has curious properties especially if the period $d+1$ of $(\alpha)_{d+1}$ is greater than $g+1$. Then $4\omega = m_b^2$ by [8, Theorem 5.2 and Corollary 6.7]. On the one hand the graph of $|b|^2 - m_b^2$ on e_k touches the zero line from below exactly $(d+1)\nu_E(e_k)$ times [8, Corollary 4.5 and Theorem 5.2]. On the other hand it has exactly one local maximum $x_k\omega - m_b^2 > 0$ at each complementary interval l_k of E [8, Lemma 4.8]. Notice that x_k may be arbitrarily large. Since for large x_k the numbers r_k are small the inequality (27) clearly forces $\omega = C(E)^{2d+2}$ to be small too.

Theorem 5.4. *A compact E is an essential support of a measure with periodic Verblunsky parameters if and only if $E = \phi^{-1}(\mathbb{T})$ for a rationally placed $\mathbb{G}$.*

Proof. By [8, Theorem 1.1] a compact $E = \cup_{j=1}^{g+1} e_j$ is essentially periodic if and only if it is rational and the phases λ_k of E are unitary rational. By the definition of ϕ_E this is equivalent to saying that $\mathbb{G}$ is rationally placed. $\square$

By Corollary 4.9 proper compact subsets $E = \cup_{k=1}^{g+1} e_k$ of $\mathbb{T}$ are in one-to-one correspondence with slits discs $\mathbb{G}_E$ by the conformal mappings $\phi = \phi_E$.

Corollary 5.5. *For any pair $g \leqslant d$ of non-negative integers there is a compact $E \subset \mathbb{T}$ consisting of $g+1$ connected components which is the essential support of a periodic measure of period $d+1$.*

Proof. We apply Theorem 5.3. If $g = d$, then we put $\alpha_j = (g+1)^{-1}$ for $j = 1, \ldots, g+1$. If $g < d$, then $\alpha_j = (d+1)^{-1}$ for $j = 1, \ldots, g$, and $\alpha_{g+1} = (d-g)(g+1)$. For an arbitrary choice of r_k in $(0,1)$ we obtain a monic polynomial b with roots on $\mathbb{T}$ such that $\{t \in \mathbb{T} : |b|^2 - 4\omega\}$ is a rotation of some rational balance compact E. By [8, Theorem 1.4] it is an essential support of some periodic measure of period $P(E)$ which obviously equals $d+1$. $\square$

Theorem 5.6. *A finite union $E = \cup_{k=1}^{g+1} e_k$ of closed non-intersecting arcs of $\mathbb{T}$ is rational if and only if its conformal mapping ϕ_E is an algebraic function.*

Proof. If E is rational, then it can be rotated to support a periodic measure and ϕ_E is algebraic by (12). If ϕ_E is algebraic, then Φ_E is algebraic by (29). Hence $\Phi_E(0)$ takes only a finite number of values [3], which are listed as

$$\Phi_1(0)\exp\left\{2\pi\mathrm{i}\sum_{k=1}^{g+1} n_k \nu_E(e_k)\right\};$$

see [8, (64)]. This is possible for arbitrary integers n_k only if all harmonic measures $\nu_E(e_k)$ are rational. $\square$

6. Approximation by rational compacts

In this section we combine the techniques of slit domains with Carathéodory's theory to approximate any compacts on $\mathbb{T}$ with a finite number of closed connected components by rational compacts. We present here Carathéodory's theory (see [12, vol III, Ch. 2]), adapted to our case.

Given a sequence of simply connected domains $\{G_n\}_{n\geqslant 1}$ containing a fixed disc $\mathbb{D}_\varepsilon = \{z : |z| < \varepsilon\}$ for some $\varepsilon > 0$ let $\mathcal{G}$ be the set of all points $z \in \mathbb{C}$ such that

$$\Delta(z) \subset \bigcap_{k=n}^{\infty} G_k \tag{36}$$

for some open disc $\Delta(z)$ centered at z and $n = n(z)$. Then the *Carathéodory kernel* $G = \mathbf{kernel}(G_n)$ of $\{G_n\}_{n\geqslant 1}$ is the connected component of $\mathcal{G}$ which contains $\mathbb{D}_\varepsilon$.

Definition 6.1. Let $G = \mathbf{kernel}(G_n)$. Then $\{G_n\}_{n\geqslant 1}$ is said to converge to G if $G = \mathbf{kernel}(G_{n_k})$ for any subsequence $\{n_k\}_{k\geqslant 1}$.

Theorem 6.1 (*Carathéodory Mapping Theorem*). *Let $\{G_n\}_{n\geqslant 1}$ be a sequence of simply connected domains inside a fixed disc $\mathbb{D}_R$ such that $\mathbb{D}_\varepsilon \subset G_n$ for every $n \geqslant 1$. Let $w = f_n(z)$ be the standard conformal mappings $f_n : G_n \to \mathbb{D}$, $f_n(0) = 0$, $f_n'(0) > 0$, of G_n onto $\mathbb{D}$, and $z = \phi_n(w) = f_n^{-1}(w)$ be their inverse mappings. If $\{G_n\}_{n\geqslant 1}$ converges to G, then*

$$\lim_n f_n(z) = f(z)$$

uniformly on compact subsets of G and

$$\lim_n \phi_n(w) = \phi(w)$$

uniformly on compact subsets of $\mathbb{D}$. The function ϕ is the standard conformal mapping of $\mathbb{D}$ onto G and $f = \phi^{-1}$.

We apply the Carathéodory mapping theorem to a very special class of domains. Namely, let $\mathbb{G}$ be a fixed slit domain with $g+1$ slits $\mathfrak{l}_k = [r_k\lambda_k, \lambda_k]$, $0 < r_k < 1$, and $\{\lambda_k^{(n)}\}_{n\geqslant 1}$ be $g+1$ sequences in $\mathbb{T}$ such that

$$\lambda_k = \lim_n \lambda_k^{(n)}, \quad k = 1, \ldots, g+1.$$

We denote by G_n the slit disc with $g+1$ slits $\mathfrak{l}_k^{(n)} = [r_k\lambda_k^{(n)}, \lambda_k^{(n)}]$.

Lemma 6.2. *The sequence G_n converges to $\mathbb{G} = \mathbf{kernel}(G_n)$.*

Proof. To begin with $\mathbb{D}_\varepsilon \subset G_n$ for $0 < \varepsilon < \min r_k$. Next, if $z \in \mathbb{G}$ then z is on the positive distance from the boundary $\partial\mathbb{G}$ of $\mathbb{G}$ implying that a small disc $\Delta(z)$ centered at z belongs to all G_n except for at most a finite number of G_n with slits passing through $\Delta(z)$. Finally, if $z \in \mathfrak{l}_k$ or $z \in \mathfrak{e}_k$, then every disc $\Delta(z)$ centered at z has a non-empty intersection with $\mathbb{C} \setminus G_n$ if $n \geqslant n(z)$. $\square$

Theorem 6.3. *For any finite union $E = \cup_{k=1}^{g+1} e_k$ of closed non-intersecting arcs e_k on $\mathbb{T}$ and any $\epsilon > 0$ there is a rational balanced compact $E^* = \cup_{k=1}^{g+1} e_k^*$ such that the Lebesgue measure $|E \triangle E^*|$ of the symmetric difference $E \triangle E^*$ is smaller than ϵ.*

Proof. By Theorem 4.2 the standard conformal mapping ϕ_E maps $\mathbb{D}$ conformally onto the slit disc $\mathbb{G}_E$. The slits $\mathfrak{l}_k$ of $\mathbb{G}_E$ have the lengths $1 - r_k$ and are placed on the rays from 0 to λ_k. For every integer n there are a sufficiently big prime p_n and $g + 1$ roots $\zeta = \lambda_k^{(n)}$, $k = 1, \dots, g + 1$, of the equation $\zeta^{p_n} = 1$ such that

$$|\lambda_k^{(n)} - \lambda_k| < 1/n, \quad k = 1, \dots, g + 1.$$

Let $\mathbb{G}^{(n)}$ be the slit disc with the radial slits $\mathfrak{l}_k^{(n)} = [r_k\lambda_k^{(n)}, \lambda_k^{(n)}]$. By Lemma 6.2 the sequence of simply connected domains $\mathbb{G}^{(n)}$ converges to $\mathbb{G}_E$ in the sense of Carathéodory. By Theorem 6.1 the sequence of the standard conformal mappings $f_n = \phi_n^{-1}$ of $\mathbb{G}^{(n)}$ onto $\mathbb{D}$ converges to $f = \phi^{-1}$ uniformly on the compact subsets of $\mathbb{G}_E = \mathbf{kernel}(\mathbb{G}^{(n)})$. Hence by Cauchy's formula $\lim_n f_n'(0) = f'(0) > 0$. By Theorem 4.3 there exists $\delta > 0$ such that

$$\mathbb{D}_\delta \subset f_n(\mathbb{D}_\epsilon) \text{ for every } n. \tag{37}$$

Let $t \in \mathbb{T}$ be an interior point of an arc $\mathfrak{e}_k$ and $\Delta(t)$ be a small open disc centered at t such that $\Delta(t) \cap \mathbb{T} \subset \mathfrak{e}_k$. Since $\mathfrak{e}_k = [\lambda_k, \lambda_{k+1}]$ and $\lim_n \lambda_k^{(n)} = \lambda_k$ and $\lim_n \lambda_{k+1}^{(n)} = \lambda_{k+1}$ there is an integer $N(\Delta)$ depending on $\Delta(t)$ such that every function f_n with $n > N(\Delta)$ extends through $\Delta(t) \cap \mathbb{T}$ to $\{z : |z| > 1\}$ by the Schwarz reflection principle:

$$f_n(z) = \frac{1}{\overline{f_n(1/\bar{z})}}.$$

Since all f_n are one-to-one mappings of $\mathbb{G}^{(n)}$ the inclusion (37) implies that the family $\{f_n\}_{n \geqslant n(t)}$ of analytic functions f_n is uniformly bounded in $\Delta(t)$. Since $\lim_n f_n(z) = f(z)$ uniformly on compact subsets of $\Delta(t) \cap \mathbb{D}$ by Theorem 6.1, we see that $\lim_n f_n(t) = f(t)$ uniformly on compact sub-arcs contained in the interior of $\mathfrak{e}_k$.

Let $t \in \mathbb{D}$ be the interior point of the slit $\mathfrak{l}_k$ and $\Delta(t)$ be a small disc centered at t which does not contain the end-points of $\mathfrak{l}_k$. If one looks in the direction of the unit circle along $\mathfrak{l}_k$ then the slit has a left side and a right side. By the Schwarz reflection principle every f_n extends through $\mathfrak{l}_k^{(n)}$ in the direction from right to left and in the opposite direction via the formula

$$f_n(z) = \frac{1}{\overline{f_n(z^*)}},$$

where z and z^* are symmetric with respect to the slit $\mathfrak{l}_k^{(n)}$. Using the same compactness arguments as in the case of $\mathfrak{e}_k$, we obtain that uniformly on $\Delta(t)$ the left boundary values of f_n on $\mathfrak{l}_k^{(n)}$ converge to the left boundary values of f on $\mathfrak{l}_k$. The same is true for the right boundary values.

It follows that the arcs $l_k^{(n)} = f_n(\mathfrak{l}_k^{(n)})$ of $\mathbb{T}$ approach the open arc l_k, whereas the open arcs $e_k^{(n)} = f_n(\mathfrak{Int}(\mathfrak{e}_k^{(n)}))$ approach the interior of e_k on $\mathbb{T}$. Since

$$\mathbb{T} = \bigcup_{k=1}^{g+1} l_k \cup \bigcup_{k=1}^{g+1} e_k = \bigcup_{k=1}^{g+1} l_k^{(n)} \cup \bigcup_{k=1}^{g+1} e_k^{(n)}$$

we see that $\lim_n |E \triangle E^{(n)}| = 0$, where $E^{(n)} = \cup_{k=1}^{g+1} e_k^{(n)}$. Every compact $E^{(n)}$ is rational since λ_k^n are the roots of the equation $\zeta^{p_n} = 1$. The compact is balanced since the p_n are chosen to be prime integers. □

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